INTERNATIONAL CONFERENCE ON INTERDISCIPLINARY SCIENCE
Volume 01, Issue 02, 2024

SHTOLS TEOREMASI VA UNING TATBIQLARI

Abdullayev Humoyiddin Baxtiyor o'g'li
Fizika-Matematika fakulteti
Matematika ta'lim yo'nalishi 3-kurs

Ilmiy rahbar: Bozorov Jo'rabek Tog'aymurodovich
Fizika-matematika fanlari bo'yicha falsafa doktori.

Annotatsiya: Biz bu maqolada Shtols teoremasini isbotini va murakab ketma-ketliklarni limitini topish. **OMOUS-2023** olimpiadasiga taklif qilingan misollarda tadbiqini ko'rib o'tamiz.

Kalit so'zlar: Shtlos teoremasi, Shtols teoremasining isboti, misollarga tadbiqi.

Teorema(Shtols). Bizga ikkita $(a_n)_{n\geq 1}$ va $(b_n)_{n\geq 1}$ ketma-ketliklar berilgan bo'lsin:

1) $(b_n)_{n\geq 1}$ ketma-ketlik qat'iy o'suvchi va

$$\lim_{n\to\infty} b_n = \infty$$

bo'lsin;

2) Quyidagi limit mavjud bo'lsin

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l$$

U holda quyidagi ketma-ketlik yaqinlashuvchi va

$$\left\{\frac{a_n}{b_n}\right\}$$

uning limiti l ga teng, ya'ni

$$\lim_{n\to\infty}\frac{a_n}{b_n}=l$$

Isbot. Teorema shartida $\{b_n\}$ ketma-ketlik qat'iy o'suvchi va limiti ∞ ga teng demak ketma-ketligimiz biror joydan($n=n_0$ chi hadidan boshlab) musbat qiymat qabul qilib boshlaydi, va

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=l$$

limit mavjud.

$\forall\varepsilon>0$ berilganda ham $\exists m\in N$ mavjudki $\forall n\geq m$ *natural sonlar uchun*

$$\left|\frac{a_{n+1}-a_n}{b_{n+1}-b_n}-l\right|<\varepsilon$$

bo'ladi. Bundan quyidagini yozib olamiz

$$(l-\varepsilon)(b_{n+1}-b_n)<a_{n+1}-a_n<(l+\varepsilon)(b_{n+1}-b_n)$$

Endi n ni k bilan, k+1 bilan, k+2 bilan va hokazo n-1 bilan almashtirib yozamiz

$$(l-\varepsilon)(b_{k+1}-b_k) < a_{k+1}-a_k < (l+\varepsilon)(b_{k+1}-b_k)$$
$$(l-\varepsilon)(b_{k+2}-b_{k+1}) < a_{k+2}-a_{k+1} < (l+\varepsilon)(b_{k+2}-b_{k+1})$$
$$(l-\varepsilon)(b_{k+3}-b_{k+2}) < a_{k+3}-a_{k+2} < (l+\varepsilon)(b_{k+3}-b_{k+2})$$

....

$$(l-\varepsilon)(b_n-b_{n-1}) < a_n-a_{n-1} < (l+\varepsilon)(b_n-b_{n-1})$$

teng buladi. Bu ifodalarni qo'shib yuborsak

$$(l-\varepsilon)(b_n-b_k) < a_n-a_k < (l+\varepsilon)(b_n-b_k)$$

Bu ifodani b_n ga bo'lamiz

$$(l-\varepsilon)(1-\frac{b_k}{b_n}) < \frac{a_n}{b_n}-\frac{a_k}{b_n} < (l+\varepsilon)(1-\frac{b_k}{b_n})$$

$$l-\varepsilon+\frac{a_k+(\varepsilon-l)b_k}{b_n} < \frac{a_n}{b_n} < l+\varepsilon+\frac{a_k-(l+\varepsilon)b_k}{b_n}$$

bilamizki

$$\lim_{n\to\infty}\frac{a_k+(\varepsilon-l)b_k}{b_n} = \lim_{n\to\infty}\frac{a_k-(l+\varepsilon)b_k}{b_n} = 0$$

yuqoridagi $\forall\varepsilon>0$ ko'ra $\exists k\in N$ natural son mavjudki barcha $n \geq k$ *uchun quyidagilar o'rinli*.

$$-\varepsilon < \frac{a_k + (\varepsilon - l)\,\mathrm{b}_k}{b_n} < \varepsilon$$

$$-\varepsilon < \frac{a_k - (\varepsilon + l)\,\mathrm{b}_k}{b_n} < \varepsilon$$

Endi p = max{m,p} deb olsak $\forall n \geq p$ natural sonlar uchun yuqoridagi ikkita tengsizligimiz bir vaqtda bajariladi. Quyidagiga egamiz

$$l - 2\varepsilon < l - \varepsilon + \frac{a_k + (\varepsilon - l)\,\mathrm{b}_k}{b_n} < \frac{a_n}{b_n} < l + \varepsilon + \frac{a_k - (l + \varepsilon) b_k}{b_n} < l + 2\varepsilon$$

$$l - 2\varepsilon < \frac{a_n}{b_n} < l + 2\varepsilon$$

$$\left| \frac{a_n}{b_n} - l \right| < 2\varepsilon$$

$\varepsilon\ soning\ ixtiyorligidan\ 2\varepsilon\ ixtiyor\ musbat\ son\ bo'ladi. Ketma -$

ketlik limit ta'rifidan

$$\left\{ \frac{a_n}{b_n} \right\}$$

ketma-ketlik yaqinlashuvchi va limiti $\boldsymbol{l}$ ga teng.

Teorema to'liq isbotlandi.

Endi teoremani olimpiada misollariga tatbiq etamiz!

1-misol(OMOUS-2023). Bizga $\{a_n\}$ ketma-ketlik berilgan bo'lib $a_0=2023$

$$a_{n+1}=\sqrt{(a_n)^2+a_n-\ln(a_n)}$$

ixtiyoriy natural son uchun o'rinli bo'lsa, quyida ketma-ketlikning limiti mavjudligini isbotlang va limitini toping:

$$\lim_{n\to\infty}\frac{a_n}{n}$$

Yechilishi: Bilamizki ln(x) < x tengsizlik ixtiyoriy x > 0 uchun to'g'ri. Bundan quyidagini topamiz:

$$a_{n+1}=\sqrt{(a_n)^2+a_n-\ln(a_n)}>\sqrt{(a_n)^2+a_n-a_n}=a_n$$

Demak $\{a_n\}$ ketma-ketlik o'suvchi va barcha hadlari musbat. Monoton ketma-ketliklarning limiti haqidagi teoremaga ko'ra $\{a_n\}$ ketma-ketlik har doim limitga ega(chekli yoki cheksiz).

Faraz qilaylik $\{a_n\}$ ketma-ketlikning limiti chekli bo'lsin, u holda

$$\lim_{n\to\infty}(a_n)^2=\lim_{n\to\infty}\left((a_n)^2+a_n-\ln(a_n)\right)=\lim_{n\to\infty}(a_n)^2+\lim_{n\to\infty}(a_n)-\lim_{n\to\infty}\ln(a_n)$$

$$\lim_{n\to\infty}(a_n)=A$$

Desak

$$A^2 = A^2 + A - \ln(A) \Rightarrow \mathrm{A} = \ln(\mathrm{A})$$

kelib chiqadi. Bundan $\{a_n\}$ ketma-ketlik limiti chekli emasligi kelib chiqadi (chunki A = ln(A) tenglik hech bir chekli A > 0 lar uchun o'rinli emas). Demak

$$\lim_{n\to\infty}\left(a_n\right) = \infty \quad .$$

Endi quyidagi ketma-ketlikni qaraylik,

$$\left\{\frac{a_n}{n}\right\}$$

bu ketma-ketlikda maxrajdagi ketma-ketlik qat'iy o'suvchi va limiti cheksizga teng ekanligidan Shtols teoremasini qo'llab,

$$\lim_{n\to\infty}\frac{a_n}{n} = \lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)-\mathrm{n}} = \lim_{n\to\infty}\left(a_{n+1}-a_n\right) = \lim_{n\to\infty}\left(\sqrt{\left(a_n\right)^2 + a_n - \ln\left(a_n\right)} - a_n\right) =$$

$$= \lim_{n\to\infty}\frac{a_n - \ln(a_n)}{\sqrt{\left(a_n\right)^2 + a_n - \ln\left(a_n\right)} + a_n} = \lim_{n\to\infty}\frac{1 - \frac{\ln(a_n)}{a_n}}{\sqrt{1 + \frac{1}{a_n} - \frac{\ln(a_n)}{a_n}} + 1} = \frac{1}{2}$$

ketma-ketlik yaqinlashuvchi va limit 0,5 ekanini topamiz.

2-Misol. Agar $\{(x_n)^4$ ketma ketlik quyidagicha aniqlangan bo'lib, n ≥ 0 da

$$x_{n+1} = \sin(x_n) \qquad (1)$$

$X_0 \in (0, \pi)$ bo'lsa, quyidagi limitni hisoblang:

$$\lim_{n\to\infty}\sqrt{n}\sin(x_n) \qquad (2)$$

Yechilishi: Agar $0 < x < \pi$ bo'lganda $0 < \sin(x) < x$ ekanligidan foydalanib

$$x_{n+1} = \sin(x_n) < x_n$$

ekanligini ya'ni $\{x_n\}$ ketma ketlik kamayuvchu va quyidan chegaralangan ekanligini topamiz. Demak $\{x_n\}$ ketma ketlik yaqinlashuvchi va uning limitini a desak (1) ifodadan limitga o'tib quyidagini topamiz:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\sin(x_n) \Rightarrow \mathrm{a} = \sin(\mathrm{a}) \Rightarrow \mathrm{a} = 0$$

Endi $\{\sqrt{n}\sin(x_n)\}$ ketma –ketlik Shtols teoremasini barcha shartlarini qanoatlantirishini e'tiborga olib (2) limitni hisoblaymiz:

$$\lim_{n\to\infty}\sqrt{n}\sin(x_n) = \sqrt{\lim_{n\to\infty}\frac{n}{\dfrac{1}{sin^2(x_n)}}} =$$

$$= \sqrt{\lim_{n\to\infty}\frac{1}{\left(\dfrac{1}{(x_n)^2} - \dfrac{1}{sin^2(x_n)}\right)}} = \sqrt{\lim_{n\to\infty}\frac{(x_n)^2 sin^2(x_n)}{(x_n)^2 - sin^2(x_n)}} =$$

$$=\sqrt{\lim_{n\to\infty}\frac{(x_n)^2\sin^2(x_n)}{(x_n)^2-\sin^2(x_n)}}=\sqrt{\lim_{n\to\infty}\frac{(x_n)^2\frac{1}{2}(1-\cos(2x_n))}{(x_n)^2-\frac{1}{2}(1-\cos(2x_n))}}=$$

$$=\sqrt{\lim_{n\to\infty}\frac{(x_n)^2\left(1-1+\frac{(2x_n)^2}{2!}-\frac{(2x_n)^4}{4!}+\dots\right)}{2(x_n)^2-\left(1-1+\frac{(2x_n)^2}{2!}-\frac{(2x_n)^4}{4!}+\dots\right)}}=\{y=x_n\}=$$

$$=\sqrt{\lim_{y\to 0}\frac{y^2\left(2y^2-\frac{(2y)^4}{4!}+\dots\right)}{\frac{16}{4!}y^4-\frac{64}{6!}y^6+\dots}}=\sqrt{3}$$

Demak

$$\lim_{n\to\infty}\sqrt{n}\sin(x_n)=\sqrt{3}$$

teng bo'ladi.

FOYDALANILGAN ADABIYOTLAR

1. T.Azlariv X.Mansurov Matematik Analiz 2-qisim toshkent “O’zbekiston “ 1995-yil.
2. Б.П.Демитович Сборник задач и упреажнений по математичискомй анализу Маскуа “наука” 1990-год
3. G.M.Fixtemgolvis matematik analiz asoslari 2-qisim Toshkent “O’zbekiston “ nashriyoti 2-qisim 1972-yil .
4. Берман Г.Н. Сборник задач по курсу математического анализа.- М.: Наука, 1985.
5. Бруй И.Н., Гаврилюк А.В и др. Лабораторный практикум по математическому анализу. - Минск.: Вышей школа, 1991