

VARIOUS MODELS OF THE LOBACHEVSKY PLANE AND THEIR GEOMETRIC INTERPRETATIONS

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Abstract: This article explores the fundamental models of the Lobachevsky plane, a cornerstone of hyperbolic geometry. It focuses on three major representations: the Poincaré disk model, the Beltrami-Klein model, and the upper half-plane model. The models are compared in terms of their geometric features, such as distance, angles, and parallel lines. We provide mathematical formulations, visual illustrations, and discuss their implications in both pure mathematics and theoretical physics.

Keywords: Lobachevsky geometry, hyperbolic plane, Poincaré disk, Beltrami-Klein model, hyperbolic lines, non-Euclidean geometry

INTRODUCTION

The discovery of non-Euclidean geometry in the 19th century marked a revolutionary turning point in the history of mathematics. Among the pioneers of this transformation was Nikolai Ivanovich Lobachevsky, who proposed a consistent geometric system that violated Euclid's fifth postulate—the parallel postulate. In Lobachevsky's hyperbolic geometry, for a given line and a point not on it, there exist infinitely many lines through the point that do not intersect the given line. This radical idea gave birth to an entirely new conception of space, one that is negatively curved and infinite in extent.

While Euclidean geometry dominates the physical and intuitive world we are accustomed to, hyperbolic geometry provides the framework for modeling curved spaces and understanding phenomena beyond the limits of classical intuition. For example, it plays a vital role in relativity theory, complex analysis, topology, and modern cosmology. The challenge, however, lies in the visualization and manipulation of hyperbolic spaces, as they cannot be embedded in Euclidean space without distortion.

To address this challenge, mathematicians have developed multiple models of the hyperbolic (Lobachevsky) plane. Each model offers a different way to represent and explore the geometry of the hyperbolic plane within the confines of Euclidean space. The three most prominent models are:

- The Poincaré disk model: A conformal model that preserves angles but distorts distances and lines.
- The Beltrami-Klein model: A projective model that preserves straight lines but not angles.
- The upper half-plane model: A conformal model used extensively in complex analysis.

These models are not merely mathematical curiosities; they provide essential tools for exploring the structure of hyperbolic geometry both theoretically and in practical applications. Each model retains the axioms and logical consistency of Lobachevsky's geometry but offers different visual and algebraic advantages depending on the context.

In this paper, we systematically examine these three fundamental models. We provide formal definitions, metric properties, and visual representations, and we compare their utility in mathematical

reasoning. This comparative study is aimed at deepening the understanding of hyperbolic geometry and its real-world implications.

METHODS

In this study, we adopt a comparative and analytical approach to examine three classical models of the Lobachevsky plane: the Poincaré disk model, the Beltrami-Klein model, and the Upper half-plane model. Our methodology involves four key components for each model: definition of the model's domain, characterization of lines, formulation of the metric (distance function), and analysis of geometric properties such as angle preservation and parallelism.

Domain Definitions

Each model defines the hyperbolic plane as a subset of Euclidean space with unique geometric rules:

Poincaré disk model (\mathbb{D}): $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

Beltrami-Klein model (\mathbb{D} again, different interpretation): Uses the same unit disk as the Poincaré model, but lines are interpreted differently.

Upper half-plane model (\mathbb{H}): $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$

Line representations

We analyze how “lines” (the geodesics of hyperbolic geometry) are constructed in each model:

-Poincaré disk: Circular arcs orthogonal to the unit circle or diameters.

-Beltrami-Klein: Straight-line chords within the disk.

-Upper half-plane: Vertical lines or semicircles orthogonal to the x-axis.

These interpretations reflect the same hyperbolic concept of a straight line but exhibit different visual and computational behaviors.

Hyperbolic metrics

We implement and compare the metric functions that define distance in each model:

Poincaré disk metric:

$$d_D(A, B) = \text{ch}^{-1} \left(1 + \frac{2|A - B|^2}{(1 - |A|^2)(1 - |B|^2)} \right)$$

Beltrami-Klein metric: Based on the projective cross-ratio:

$$d_{BK}(A, B) = \frac{1}{2} \ln \left(\frac{|QA||PB|}{|PA||QB|} \right)$$

where P and Q are the ideal points on the boundary.

Upper half-plane metric:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and the distance function is:

$$d_H(A, B) = \text{ch}^{-1} \left(1 + \frac{|A - B|^2}{2y_A y_B} \right)$$

These metrics are central to understanding how distance and curvature behave in hyperbolic space.

Comparison strategy

To ensure a comprehensive analysis, we compare the models using the following criteria:

- Conformality (angle preservation)
- Geodesic structure
- Computational efficiency
- Suitability for complex analysis or physics

Throughout this study, we employ visual diagrams and analytic computations to demonstrate equivalence and highlight differences. We also explore how these models transform under Möbius transformations, which preserve hyperbolic structure.

RESULTS

Our comparative study of the three classical models of the Lobachevsky plane reveals fundamental insights into the geometric, analytic, and visual properties of hyperbolic geometry. Each model provides a consistent realization of hyperbolic space but emphasizes different features of its structure. Below, we summarize the main findings from our analysis.

Poincaré disk model

The Poincaré disk model excels in preserving angles due to its conformal nature. This makes it highly suitable for complex analysis and visualization of isometries. Geodesics in this model appear as circular arcs orthogonal to the boundary circle, which maintains visual elegance and mathematical consistency. For instance, the angle between two intersecting geodesics matches the Euclidean angle at their intersection point.

Advantages: preserves angles (conformal), supports Möbius transformations, visually intuitive for many geometric constructions

Distance increases rapidly as one approaches the boundary. A point near the edge is infinitely far from the center in the hyperbolic metric, despite being finitely close in Euclidean terms.

Beltrami-Klein model

The Beltrami-Klein model represents geodesics as Euclidean chords, which are straight lines in the disk. This makes it ideal for linear constructions and applications in projective geometry. However, it is not conformal, meaning angles between intersecting lines are distorted. The model's usefulness lies in its intuitive representation of lines and simplicity in drawing hyperbolic polygons.

Advantages: geodesics are straight lines (projective simplicity), suitable for convexity analysis and visualization of shortest paths, emphasizes the projective structure of hyperbolic geometry.

Drawback: angles are not preserved; not suitable where conformality is required.

Upper half-plane model

The upper half-plane model is especially important in complex analysis, as it naturally accommodates Möbius transformations of the form:

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

which preserve hyperbolic structure. Geodesics appear as vertical lines or semicircles orthogonal to the real axis. The model is highly flexible and often used in number theory (e.g., modular forms).

Advantages: conformal model; preserves angles, natural compatibility with complex functions, useful in theoretical physics and automorphic function theory.

Metric behavior: Distance between two points $A = x_1 + iy_1, B = x_2 + iy_2$ is given by:

$$d_H(A, B) = \text{ch}^{-1} \left(1 + \frac{|A - B|^2}{2y_1 y_2} \right)$$

Summary table of model properties

Property	Poincaré Disk	Beltrami-Klein	Upper Half-Plane
Conformal (angle-preserving)	Yes	No	Yes
Geodesic shape	Arcs orthogonal to boundary	Straight chords	Semicircles / vertical lines
Euclidean straight lines	No	Yes	No
Metric simplicity	Moderate	Complex (cross-ratio)	Simple (hyperbolic fraction)
Application areas	Visualization, complex analysis	Projective geometry	Complex analysis, number theory

These results confirm that while all three models are mathematically equivalent representations of the Lobachevsky plane, each brings a unique perspective and utility, making them suitable for different types of mathematical and scientific investigations.

DISCUSSION

The comparative analysis of the Lobachevsky plane models highlights the diverse ways in which hyperbolic geometry can be visualized, interpreted, and applied. While all three models are mathematically equivalent—in the sense that there exist isometries mapping one to another—they emphasize different geometric structures and serve distinct purposes in mathematical practice and scientific applications.

Conceptual implications

The differences in angle preservation, geodesic representation, and metric formulation are not merely technical nuances; they fundamentally shape how mathematicians conceptualize hyperbolic space. For instance:

- The Poincaré disk model offers a visually elegant and conformal representation that closely mimics Euclidean intuition, making it especially useful for teaching and graphical demonstrations.
- The Beltrami-Klein model aligns with projective geometry, and while it sacrifices conformality, it provides powerful tools for studying convexity, shortest paths, and dual structures.
- The Upper half-plane model directly connects to complex analysis, particularly through Möbius transformations and the action of modular groups such as $PSL(2, Z)$, which are central in number theory and mathematical physics.

Practical applications

Each model’s strengths lend themselves to specific domains:

- In computer graphics, the Beltrami-Klein model is often favored due to its straight-line geodesics, which simplify rendering and collision detection in hyperbolic virtual spaces.
- In complex analysis and modular form theory, the upper half-plane model is indispensable due to its natural embedding in the extended complex plane.

-In educational settings, the Poincaré disk model remains popular for its intuitive depiction of hyperbolic angles and triangles.

-Furthermore, these models are extensively used in hyperbolic tiling, knot theory, non-Euclidean navigation algorithms, and general relativity, where negative curvature spaces play a pivotal role.

Equivalence and isometries

Though each model appears visually distinct, they are all **isometric representations** of the same underlying structure. For example, an isometry between the upper half-plane H and the Poincaré disk D is given by the Möbius transformation:

$$z \mapsto \frac{z - i}{z + i}$$

This transformation preserves hyperbolic distances and maps vertical lines or semicircles in H to circular arcs in D . Such mappings are crucial in translating results between models, especially in theoretical research.

Limitations and challenges

While powerful, each model has limitations:

- The Beltrami-Klein model distorts angles, making it unsuitable for problems involving precise angle measurements.
- The Poincaré disk model, while conformal, distorts Euclidean distances significantly near the boundary, complicating certain physical interpretations.
- The upper half-plane model, despite its analytical strengths, can be unintuitive for purely geometric or visual analysis.

These trade-offs suggest that the choice of model should be context-dependent, guided by the goals of the study-be it visual clarity, computational ease, or analytic precision.

CONCLUSION

This study has provided a comprehensive examination of the three classical models of the Lobachevsky plane-namely, the Poincaré disk model, the Beltrami-Klein model, and the upper half-plane model-each offering a unique lens through which to explore hyperbolic geometry.

While mathematically isometric and equivalent, the models differ significantly in their visual representations, geometric properties, and applicability:

- The Poincaré disk model excels in angle preservation and is visually intuitive, making it ideal for pedagogical purposes and theoretical visualizations.
- The Beltrami-Klein model, with its straight-line geodesics, is optimal for projective constructions and practical applications like computational geometry and hyperbolic embeddings.
- The upper half-plane model, rooted in complex analysis, bridges geometry with algebraic structures, and is widely used in number theory, modular forms, and theoretical physics.

Our results reaffirm that no single model is universally superior; rather, their utility is context-dependent. Understanding their interrelationships via Möbius transformations and isometries enables mathematicians and scientists to translate findings across models and leverage their respective strengths for diverse applications.

Moreover, the exploration of these models has implications beyond pure mathematics, extending into areas such as quantum field theory, relativity, data visualization, and virtual reality. As hyperbolic

geometry continues to find new applications in topology, network theory, and machine learning, these models will remain essential tools for theoretical development and computational implementation.

Future research could explore:

-Generalizations to higher dimensions (e.g., hyperbolic 3-space models);

-Dynamic visualizations and interactive simulations of model transformations;

Applications in hyperbolic neural networks and non-Euclidean data structures;

-The role of these models in modern physical theories, such as string theory or holographic duality

By deepening our understanding of the Lobachevsky plane through these foundational models, we pave the way for both theoretical advancements and practical innovations in diverse scientific disciplines.

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