

ISOMETRIES OF \log – ALGEBRAS

Madaminov B.A.

(Doktor of Physical and Mathematical sciences (Phd). Non-government educational institution "Mamun University")

Boltaboyeva G.Z.

(1st year student in the field of "Accounting and Auditing" at Mamun University)

Rustamboyeva I.I.

(1st year student in the field of "Accounting and Auditing" at Mamun University)

Abstract.: In this paper studied isometries of F –spaces of integrable functions with logarithm. In particular, using passports of Boolean algebra, a necessary and sufficient condition of isometry F –spaces of integrable functions of logarithm with respect to strictly positive σ –finite measures is proved. In this work generalized \log – algebras are considered separately.

Keywords: Functional spaces; boolean algebras; complete boolean algebras; homogeneous Boolean algebras; passport of boolean algebra; internal \log – algebras; external \log – algebras; generalized \log – algebras; isomorphisms; isometries.

Introduction: One of the important classes of Banach function spaces are the spaces $L_p(\Omega, A, \mu)$, $1 \leq p < \infty$ of all functions defined on a measurable space (Ω, A, μ) whose p –th powers are integrable with respect to a finite or σ –finite measure μ (functions coinciding almost everywhere are identified). The study of isometries of L_p –spaces was started by Banach in [1], where he described all isometries of the spaces $L_p[0,1]$, $p \neq 2$.

In [2], J.Lamperti gave a characterization of all linear isometries for L_p –spaces $L_p(\Omega, A, \mu)$, where (Ω, A, μ) is an arbitrary space with a finite measure μ . The final result in this formulation belongs to Yedan [3], who gave a complete description of all isometries between L_p –spaces associated with various measures. One of the corollaries of such descriptions of isometries in the spaces $L_p(\Omega, A, \mu)$ is

the establishment of isometry for L_p –spaces $L_p(\Omega, A, \mu)$ and $L_p(\Omega, A, \nu)$ in the case when the measures μ and ν are strictly positive finite measures.

In [4], F –spaces of logarithm-integrable functions L_{\log} were introduced, which are analogs of L_p –spaces.

Interest in these spaces is due to the fact that they are close to the class of Nevanlinna functions holomorphic in the circle and integrable with the logarithm on the boundary of the circle [5], i.e. satisfying the condition

$$L(f) := \sup_{r < 1} \frac{1}{2\pi} \int \log(1 + |f(re^{i\theta})|) d\theta < \infty \quad .$$

In this paper, we introduce generalizations of these log–spaces. They are called generalized log–spaces. For these spaces, a necessary and sufficient condition for their isometry is established for the case of σ –finite measures.

Let us give some information from [6].

Boolean algebra is a distributive structure with zero and unit, that are unequal to each other.

Boolean algebra is said to be *complete* if every set of its elements has upper and lower bounds.

A measure μ on a Boolean algebra ∇_μ is called *strictly positive* if $\mu(x) = 0$ implies that $x = 0$.

Definition 1. Let [4]

$$L_{\log}(\nabla, \mu) = \left\{ f \in L_0(\nabla, \mu) : \int_{\Omega} \log(1 + |f|) d\mu < +\infty. \right\}$$

$L_{\log}(\nabla, \mu)$ be an F –space with respect to the F –norm $\|f\|_{\log} = \int_{\Omega} \log(1 + |f|) d\mu$. In

[4], it is also shown that $L_{\log}(\nabla, \mu)$ is an algebra. We call this algebra the *external log–algebra*.

Let μ, ν be strictly positive σ –finite measures on a Boolean algebra ∇_μ ,

$\frac{d\nu}{d\mu} = h \geq 0$ is the Radon - Nikodym derivative of ν with respect to μ , i.e.

$$\int_{\Omega} f d\nu = \int_{\Omega} h f d\mu,$$

and since μ and ν are strictly positive, the support of h is equal to one. It is clear that then the external log–algebras satisfy the equalities

$$\begin{aligned} L_{\log}(\nabla_{\nu}) &= \left\{ f \in L_0(\nabla) : \int_{\Omega} \log(1 + |f|) d\nu < +\infty \right\} = \\ &= \left\{ f \in L_0(\nabla) : \int_{\Omega} h \cdot \log(1 + |f|) d\mu < +\infty \right\} = L_{\log}^{\nu}(\nabla_{\mu}), \\ \|f\|_{\log, \nu} &= \int_{\Omega} \log(1 + |f|) d\nu = \int_{\Omega} h(\log(1 + |f|)) d\mu := \|f\|_{\log, \mu}^{\nu} \end{aligned} \quad (1)$$

Consider now the following analogue of the space of log–integrable measurable functions.

Definition 2.

$$L_{\log}^{(\nu)}(\nabla_{\mu}) = \left\{ f \in L_0(\nabla) : \int_{\Omega} \log(1 + h|f|) d\mu < +\infty \right\}$$

The following assertion implies that this space is an F -space with respect to the F -norm

$$\|f\|_{\log, \mu}^{(\nu)} = \int_{\Omega} (\log(1 + h|f|)) d\mu$$

We call $L_{\log}^{(\nu)}(\nabla_{\mu})$ internal log–algebras.

RESULTS AND DISCUSSIONS

Theorem 1. For any strictly positive σ -finite measures μ and ν , the spaces $L_{\log}(\nabla_{\mu})$ and $L_{\log}^{(\nu)}(\nabla_{\mu})$ are isometric

Proof. Indeed, for the measures μ and ν the mapping $U : L_{\log}(\nabla_{\mu}) \rightarrow L_{\log}^{(\nu)}(\nabla_{\mu})$, defined by the equality $U(f) = h^{-1}f$, $f \in L_{\log}(\nabla_{\mu})$ is a linear surjective isometry from $L_{\log}(\nabla_{\mu})$ and $L_{\log}^{(\nu)}(\nabla_{\mu})$.

Theorem is proved.

Remark. As it can be seen from Lemma 1, F -spaces $L_{\log}(\nabla_{\mu})$ and $L_{\log}(\nabla_{\nu}) = L_{\log}^{(\nu)}(\nabla_{\mu})$ are generally not isometric

Let μ, ν_1 and $\nu_2 \in M(\nabla)$, $\mu(\Omega) = 1$, ν_1 and ν_2 are σ -finite. Then (1) implies that the equalities

$$\int_{\Omega} f d\nu_1 = \int_{\Omega} h_1 f d\mu, \quad (2)$$

$$\int_{\Omega} f d\nu_2 = \int_{\Omega} h_2 f d\mu, \quad (3)$$

hold.

Theorem 2. The function $\|f\|_{\log, \nu}^{v_1(v_2)} = \int_{\Omega} h_1 \log(1 + h_2 |f|) d\mu$ is the F -norm on the F -space

$$L_{\log}^{v_1(v_2)}(\nabla_{\mu}).$$

Proof.

By virtue of (1), we have

$$\|f\|_{\log, \nu}^{v_1(v_2)} = \int_{\Omega} h_1 \log(1 + h_2 |f|) d\mu = \int_{\Omega} \log(1 + h_2 |f|) d\nu_1 = \|f\|_{\log, \nu_1}^{(v_2)}$$

By virtue of Proposition 1, [[4], Lemma 2.1] the function $\|f\|_{\log, \mu}^{(v_2)}$ is the F -norm on

$$L_{\log}^{v_2}(\nabla_{\mu}), \text{ hence, } \|f\|_{\log}^{v_1(v_2)} \text{ is also } F\text{-norm on } L_{\log, \nu_1}^{v_1(v_2)}(\nabla_{\mu})$$

Let $\nu_1, \nu_2, \nu_3, \nu_4, \mu$ be strictly positive finite σ -finite measures on ∇_{μ} , $\mu(\Omega) = 1$.

References

1. Dykema K., Sukochev F., Zanin D. Algebras of log-integrable functions and operators, Journal, Complex Anal. Oper. Theory 10, (8), (2016), pp. 1775–1787.
2. Abdullaev R.Z., Chilin V. Isomorphic Classification of *-Algebras of log-Integrable Measurable Functions. Algebra, Complex Analysis, and Pluripotential Theory. USUZCAMP 2017. Springer Proceedings in Mathematics and Statistics, 264, pp.73-83. Springer, Cham.
3. Abdullaev R., Chilin V., Madaminov B. Isometric F -spaces of log-integrable function. Siberian Electronic Mathematical Reports. том 17, pp. 218-226(2020).
4. Abdullaev R.Z., Madaminov B.A. arXiv:submit/3383988 [math.FA] 24 sep 2020.
5. Vladimirov D.A. Boolean Algebras in Analysis. Mathematics and its Applications, 540, Kluwer Academic Publishers, Dordrecht (2002).