SPECTRAL ASYMPTOTICS OF DISCRETE SCHRÖDINGER OPERATORS UNDER POINT INTERACTION POTENTIALS"

Madatova Fotima Abdirakhimovnaa, PhD Student.
National University of Uzbekistan, e-mail: fotimamadatova2@gmail.com

Abstract. We investigate the one-particle discrete Schrödinger operator with Dirac delta potential on the d-dimensional lattice. We show that the operator the operator has a unique eigenvalue and obtain an asymptotics expansion for this eigenvalue as wight of potential approaches the infinity.

Keywords: Schrödinger operator, spectrum, eigenvalue, Fredholm determinant, eigenvalue asymptotics.

I. INTRODUCTION

The spectral properties of discrete Schrödinger operators have attracted significant attention in the study of both combinatorial Laplacians and quantum graphs (see [3], [4], [6], [9], [11], [15], [21], and references therein for recent summaries). In particular, the eigenvalue behavior of Schrödinger operators on lattices has been investigated in [1], [5], [8], [12], [13], and briefly discussed in [10] and [16], under the assumption that the potential is a Dirac delta function.

In addition, the spectral properties of the Hamiltonian describing the motion of a single quantum particle on a lattice in an external field were studied in [12, 16, 17, 19, 20]. These works investigated the number of eigenvalues and their locations, depending on the value of the interaction energies.

In the present paper, we study spectral properties of the discrete Schrödinger operator $\widehat{H}_\mu$, $\mu \in \mathbb{R}$, on the d- dimensional lattice with a zero-range potential (Dirac delta potential) concentrated around $x = 0 \in \mathbb{Z}^{\mathrm{d}}$ and with an extended dispersion relation depending on the real parameter β ($\beta > 0$).

In the case when the potential is δ function, the existence and uniqueness of $z(\mu)$ have been investigated in the [1, 13] and the Hamiltonian of a system of three quantum mechanical particles moving on the three-dimensional lattice $\mathbb{Z}^3$ and interacting via zero-range attractive potentials is considered in the [2, 16, 19]. In [14], asymptotical behaviour of the eigenvalues as $\mu \to 0$ was studied.

Note that the authors of [24] have investigated this operator, and they proved that the operator $\widehat{H}_\mu$ has a unique eigenvalue $z(\mu, \beta)$ for each value of the parameter μ. In addition, they showed that $z(\mu, \beta)$ is infinitely many times differentiable as the function of μ for $d = 1,2$, and established the asymptotics for this eigenvalue as $\mu \to \infty$.

In the present paper, we improve the results obtained in [24], that is, we show that $z(\mu,\beta)$ is infinitely many times differentiable as the function of μ for any dimension of the lattice $\mathbb{Z}^d$ and we get the more accurate asymptotics of eigenvalues as $\mu\to\infty$
Note that, to the best of our knowledge, this result is new in this field.
In order to facilitate the description of the content, we introduce the followius notations used throughout this manuscript: $\mathbb{Z}^d$ is the d–dimensional lattice and $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d = (-\pi,\pi]^d$ is the d-dimensional torus equipped with the Haar measure.

II. Discrete Schrödinger operator

The one-particle Schrödinger operator H_μ in the momentum representation is defined as

$$H_\mu = \mathcal{F}^*\widehat{H}_\mu\mathcal{F}, \quad H_\mu = H_0 - V_\mu,$$

where

$$H_0 = \mathcal{F}^*(-\triangle)\mathcal{F}, \quad V_\mu = \mathcal{F}^*\widehat{V}_\mu\mathcal{F}. \tag{1}$$

Here, $\mathcal{F}$ stands for the standard Fourier transformation $\mathcal{F}: L^2(\mathbb{T}^d)\to\ell^2(\mathbb{Z}^d)$ with the inverse $\mathcal{F}^*: \ell^2(\mathbb{Z}^d)\to L^2(\mathbb{T}^d)$. Explicitly, the non-perturbed operator H_0 acts on $L^2(\mathbb{T}^d)$ as a multiplication operator by the function $\varepsilon(\cdot)$:

$$(H_0 f)(p) = \varepsilon(p)f(p), \quad f\in L^2(\mathbb{T}^d), \quad p\in\mathbb{T}^d,$$

where

$$\varepsilon(p) = \sum_{j=1}^{d}(1-\cos p_j) + \beta(1-\cos 2p_j), \quad p\in\mathbb{T}^d, \quad \beta\geq 0. \tag{2}$$

In the physical literature, the function $\varepsilon(\cdot)$, being a real valued-function on $\mathbb{T}^d$, is called the dispersion relation of the Laplacian operator $-\triangle$.
The potensial operator is transformed into a rank one integral operator

$$(Vf)(p) = \frac{\mu}{(2\pi)^d}\int_{\mathbb{T}^d} f(q)dq, \quad f\in L^2(\mathbb{T}^d). \tag{3}$$

Now, we present the lemma on the range of the function $\varepsilon(p)$ defined in the form (3).
Lemma 1. ***a) For any $\beta>0$, the function $\varepsilon(p)$ has a non-degenerate minimum at the point $Q_0=(0,0,\dots,0)$ and $\varepsilon(Q_0)=0$.***

b1) If $0<\beta<\frac{1}{4}$, then the function $\varepsilon(p)$ has a non-degenerate maximum at the point $Q_1=(\pi,\pi,\dots,\pi)$ and $\varepsilon(Q_1)=2d$.

b2) When $\beta=\frac{1}{4}$, the function $\varepsilon(p)$ has degenerate maximum at the point $Q_1=(\pi,\pi,\dots,\pi)$ and $\varepsilon(Q_1)=2d$.

b3) When $\beta>\frac{1}{4}$, the fuction $\varepsilon(p)$

$Q_2 = \left(arccos\left(-\frac{1}{4\beta}\right), arccos\left(-\frac{1}{4\beta}\right), \dots, arccos\left(-\frac{1}{4\beta}\right)\right)$*has a non-degenerate maximum at the point* $\varepsilon(Q_2) = \sum_{i=1}^{d} \frac{(1+4\beta)^2}{8\beta} = \frac{(1+4\beta)^2 d}{8\beta}$.

III. The essential spectrum.

H_0 is a multiplication operator by the real valued, continuous function and the perturbation V is the one-dimensional operator and, therefore, in accordance to the Weyl theorem on the stability of the essential spectrum the equality $\sigma_{ess}(H_\mu) = \sigma_{ess}(H_0)$ holds the essential spectrum of the operator H_μ consists of the following segment on the real axis:

$$\sigma_{ess}(H_\mu) = [\varepsilon_{\min}, \varepsilon_{\max}]$$

that is

$$\varepsilon_{\min} = 0 \quad \text{and } \varepsilon_{\max} = \begin{cases} 2d, & 0 \le \beta \le \frac{1}{4}, \\ \frac{(1+4\beta)^2 d}{8\beta}, & \beta > \frac{1}{4}. \end{cases}$$

IV. Fredholm determinant of the operator H_μ

For any $\mu \in \mathbb{R}$, we define the Fredholm determinant associated with the operator H_μ as a regular function in $z \in \mathbb{R} \setminus [\varepsilon_{\min}, \varepsilon_{\max}]$ as

$$D(\mu, z) = 1 - \frac{\mu}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{dp}{\varepsilon(p) - z}. \tag{4}$$

Lemma 2. *The number* $z \in \mathbb{C} \setminus [\varepsilon_{min}, \varepsilon_{max}]$ *is an eigenvalue of* H_μ *if and only if* $D(\mu, z) = 0$ *and their multiplicities are the same.*

Lemma 3. a) Let $d = 1,2$. Then

$$I = \lim_{z \to \varepsilon_{\min}-} \int_{\mathbb{T}^d} \frac{dq}{\varepsilon(q) - z} = \int_{\mathbb{T}^d} \frac{dq}{\varepsilon(q) - \varepsilon_{\min}} = +\infty.$$

b) Let $d \ge 3$. In that case the integral

$$I = \int_{\mathbb{T}^d} \frac{dq}{\varepsilon(q) - \varepsilon_{\min}}$$

exists.

Lemma 4. *a) a) If* $d = 1,2$ *or* $d = 3,4$ *and* $\beta = \frac{1}{4}$, *then*

$$d(\varepsilon_{\max}) = \int_{\mathbb{T}^d} \frac{dq}{\varepsilon(q) - \varepsilon_{\max}} = -\infty$$

b) If $d \geq 3$ *and* $\beta \neq \frac{1}{4}$ *or* $d > 4$ *and* $\beta = \frac{1}{4}$, *then*

$$d(\varepsilon_{max}) = \int_{\mathbb{T}^d} \frac{dq}{\varepsilon(q) - \varepsilon_{\max}}$$

exists.

Lemma 5. a) For any $\mu > 0$ there exists a unique number $z(\mu, \beta)$ in the interval $(-\infty, 0)$ such that $D(\mu, z(\mu, \beta)) = 0$, and $z(\mu, \beta)$ is differentiable with respect to the μ-variable.
b) For any $\mu < 0$ there exists a unique number $z(\mu, \beta)$ in the interval $(\varepsilon_{max}, \infty)$ such that $D(\mu, z(\mu, \beta)) = 0$, and $z(\mu, \beta)$ is differentiable with respect to the μ-variable.

Theorem 1. Let $d (d \geq 3)$ *and*

$$\mu_0 = \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{dq}{\varepsilon(q) - 0}\right)^{-1}, \qquad \mu^0 = \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{dq}{\varepsilon(q) - \varepsilon_{\max}}\right)^{-1}.$$

a) If $\mu > \mu_0$, *the operator* H_μ *has a unique eigenvalue* $z(\mu, \beta)$*in the interval* $(-\infty, 0)$ *and has no eigenvalue to the right of essential spectrum.*

b) If $\mu < \mu^0$, *the operator* H_μ *has a unique eigenvalue* $z(\mu, \beta)$*in the interval* $(\varepsilon_{max}, \infty)$ *and has no eigenvalue to the left of essential spectrum.*

c) If $\mu^0 \leq \mu \leq \mu_0$, *the operator*$H_\mu$ *has no an eigenvalue eighter in the interval* $(-\infty, 0)$ *or* $(\varepsilon_{max}, +\infty)$.

Proof. Lemmas 3 and 5 imply the proof.

Lemma 6. $z(\mu, \beta)$ *satisfies the asymptotics*

$$z(\mu, \beta) = \frac{1}{-\frac{1}{\mu} - d(1+\beta)\frac{1}{\mu^2} - \frac{d((2d-1)\beta^2 + 4d\beta + (2d-1))}{2!}\frac{1}{\mu^3} + O(\frac{1}{\mu^4})},$$

as $\mu \longrightarrow \infty$.

Theorem 2. $z(\mu, \beta)$ *satisfies the asymptotics*

$$z(\mu, \beta) = -\mu + d(1+\beta) - \frac{d(1+\beta^2)}{2!}\frac{1}{\mu} - (1+\beta)\big((d-1)\beta^2 + 2d\beta + d - 1\big)\frac{1}{\mu^2}$$
$$+ \frac{d^2\big((2d-1)\beta^2 + 4d\beta + (2d-1)\big)\big((4d+1)\beta^2 + 8d\beta + 4d + 1\big)}{(2!)^2}\frac{1}{\mu^3} + O\left(\frac{1}{\mu^4}\right),$$

as $\mu \to \infty$.

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